# Successful Students' Conceptions of Mean, Standard Deviation, and The Central Limit Theorem

David Mathews
Central Michigan University
Mt. Pleasant, Michigan

Julie M. Clark
Hollins University
Roanoke, Virginia
©7/25/2007

## 1 Introduction

This paper is one in an emerging series of studies by members of the "Research in Undergraduate Mathematics Education Community," or *RUMEC*, concerning the nature and development of college students' mathematical knowledge. We present analyses of audio-taped clinical interviews with college freshmen immediately after they completed an elementary statistics course with a grade of "A". The point of these interviews was neither to see how quickly isolated facts could be recalled, nor was the point to see how little students remember. Rather, the goal was to determine as precisely as possible the conceptions of mean, standard deviation and the Central Limit Theorem which the most successful students held shortly after having completed a statistics course.

Be it in statistics or any other mathematical content area, teachers want their students to learn and retain as much mathematics as possible. But as Schlomo Vinner put it in [31]:

> There is one overall question which bothers mathematics teachers as well as all other teachers: What will remain in our students' minds after the end of the course and the final exam?

Since experienced teachers generally know (and are dissatisfied with) the answer to this question, one might well be tempted to experiment with pedagogical innovations to increase retention of knowledge. But such an approach assumes that we know what constitutes statistical knowledge. As early as the late 1970's, leaders within the mathematics education community were urging a moratorium on such studies of performance and argued for an increased research effort to help us to understand the cognitive development in the learners of mathematics ([14], [22]). Considerable progress has been made in this area with regard to functions (see [2], [12], [31] for examples of this). Moreover, a number of general epistemological frameworks have been developed (see [1], [10] , [23], [27]) and applied to a variety of mathematical content areas. Very little work, however, has been done relative to studying the cognitive development of college students with regard to statistical concepts. In an excellent survey entitled *Difficulties in Learning Basic Concepts in Probability and Statistics: Implications for Research*, Garfield and Ahlgren [8] summarized the situation and noted:

> Although many articles in the education literature recommend how to teach statistics better, there is little published literature on how students actually learn statistics concepts.

The present paper reports on a preliminary attempt to better understand the development of student knowledge of some fundamental statistical concepts.

## 1.1 The RUMEC APOS Primer

A very specific theoretical perspective is being used by members of *RUMEC* for the purpose of studying the learning of collegiate mathematics. This has developed through an attempt to understand the ideas of Piaget [19] concerning the mental activity known as *reflective abstraction* and to reformulate and extend these ideas to the context of college-level mathematics. What follows is an abbreviated description of the four specific cognitive constructions which we use to analyze student learning of mathematical concepts. In this paper, we will use these descriptions to characterize mathematical understanding at different levels as observed in clinical interviews. The mental constructions are called actions, processes, objects, and schemas, and so the theory is sometimes called the APOS Theory. [1] The lowest level of construction in APOS Theory is that of an action.

**Action.** An *action* is any transformation of objects to obtain other objects. It is perceived by the individual as being at least somewhat external to himself, as it has the characteristic that at each step, the next step is triggered by what has come before.

For example, if one is given a formula for a function and a specific point, and one then calculates the value of the function at that specific point, one is performing an action. If an individual solves a given equation using as a guide the steps in the solution for a similar equation, then he or she is performing an action. Finally, if one is given the general rule for finding the derivative of a polynomial function, and applies the rule to a specific function to find its derivative, then one is performing an action.

We say that an individual has an *action conception* of a given concept if her or his depth of understanding is limited to performing actions relative to that concept. For example, if one's understanding of standard deviation is limited to the ability to calculate the standard deviation of a particular set given the formula, then this individual would be said to have an action conception of standard deviation. It should be noted, of course, that someone with a mature understanding of a mathematical concept may well perform actions when appropriate, but such a person is not *limited* to performing actions.

Piece-wise defined functions, inverses of functions, composition of functions, sets of functions, the notion that the derivative of a function is a function, and the idea that a solution of a differential equation is a function are all sources of great difficulty for students. According to the APOS theoretical perspective, a major reason for the difficulty is that the learner is not able to go beyond an action conception of function and all of these notions require process and/or object conceptions. (See [2] for an elaboration of these ideas.)

Although an action conception of any mathematical concept is inherently very limited, the following paragraphs describe the way in which actions form the crucial beginning of deeper understanding of a concept.

**Process.** When an action is repeated, and the individual reflects upon it, it may be *interiorized* into a *process*. That is, an internal construction is made that performs the same action. In contrast to an action, a process is perceived by the individual as being internal, and under one's control, rather than as something one does in response to external cues. An individual who has a *process conception* of a transformation can reflect on, describe, or even reverse the steps of the transformation without actually performing those steps.

In the case of functions, a process conception allows the subject to think of a function as receiving one or more inputs, or values of independent variables, performing one or more operations on the inputs and returning the results as outputs, or values of dependent variables. In the context of abstract algebra, a process understanding of cosets would include the ability to think about the formation of a

[1]The reader is referred to Asiala, et al [1] for a complete discussion of each of the four cognitive constructions.

coset by operating a fixed element of the group on every element in a particular subgroup. Again, it is not necessary to perform the operations, but only to think about them being performed. Thus, with a process conception, cosets can be formed in situations where formulas are not available. (For example, see [6].)

**Object.** When an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that this entity has properties, and that transformations (whether they be actions or processes) can act on it, and he or she becomes able to actually construct such transformations, then he or she is thinking of this process as an *object*. We say that the process has been *encapsulated* into an object, and that the individual has an *object conception* of the concept. For example, a student who understands the standard deviation of a set as a measure of spread which roughly averages distances from the mean would have an object conception of standard deviation.

In the course of performing an action or process on an object, it is often necessary to *de-encapsulate* the object back to the process from which it came in order to use its properties in manipulating it. It is easy to see how encapsulating processes to objects and de-encapsulating the objects back to processes arise when one is thinking about manipulations of functions such as adding, multiplying, differentiating or forming sets of functions. In an abstract algebra context, given an element $x$ and a subgroup $H$ of a group $G$, if an individual thinks generally of the (left) coset of $H$ determined by $x$ as a process of operating with $x$ on each element of $H$, then this process can be encapsulated to an object $xH$. With such an object conception, cosets can be named and operations can be performed on them ([6]). Various actions on cosets of $H$, such as counting their number, comparing their cardinality, and checking their intersections can make sense to the individual who has an object conception of coset. The relationship between the process and object conceptions of cosets with regard to, say, checking the intersection of two cosets is as follows. Thinking about the problem of investigating the intersection involves the interpretation of cosets as objects. The actual determination of the intersection requires that these objects be de-encapsulated in the individual's mind so as to make use of the properties of the processes from which these objects came (certain kinds of set formation in this case).

So cognitive objects in APOS Theory are those things which can be acted upon and can have properties. Hence, a function is an object for a person who can view a transformation as a member of a particular function space, and the differentiation operator in calculus is an object for someone who can view this as a particular algebraic derivation.

In general, encapsulating processes into objects is considered by mathematics education researchers to be extremely difficult ([19], [23], [25]). Only individuals who are particularly mathematically talented are thought to do this "naturally" with very little intervention. [2]

**Schema.** An individual's *schema* for a certain piece of mathematics is that person's own cognitive framework which connects in some way all of the ideas that the individual either consciously or subconsciously views as related to the piece of mathematics.

For example, one may have a schema for limits which enables one to coordinate in some coherent fashion one's cognitive representations of closeness in the domain, understanding of closeness in the range, and conception of function (see [3]). Mathematical rules, such as the chain rule for differentiation, which require coordinating two or more actions, processes, or objects may also be understood via a schema.

Schemas themselves can be treated as objects and included in the organization of "higher level" schemas. When this happens, we say that the schema has been *thematized* to an object. For example, Euclidean Geometry is a thematized schema, which is an object for someone who knows several

---

[2] One defining characteristic of the *RUMEC* theoretical position is the belief that many people other than just the mathematically talented can make these advanced constructions if given proper activities and a suitable environment.

geometries, moves among them, and can compare and contrast them. According to APOS Theory, thematized schemas and encapsulated processes are the only mental constructs with object status.

Schemas are important to the individual for mathematical empowerment, but in general, we are very far from knowing all of their specifics and have not studied much about how this organization determines mathematical performance. The reader familiar with mathematics education literature will note that a schema in APOS Theory is not unlike the *concept image* described by Tall and Vinner [27] but may be more fine-grained in its description. It is apparent that different individuals might have quite different concept images or schemas for the same mathematical concept. That this is as true for professional mathematicians ([29]) as well as for beginning students is one of the profound implications of *constructivism*. [3]

As researchers, we attempt to describe a generic road map of understanding which includes individual mental constructs, their origins, and their relationships to one another. Such theoretical models are used to describe both elementary constructs and schemas and are referred to in APOS Theory as *genetic decompositions*.

## 1.2 Moving toward Advanced Mathematical Thinking

From the point of view of APOS Theory, cognitive development in an individual relative to a given concept can be explained, or modeled, as follows. First, the individual must perform actions implementing a concept. Next, these actions are interiorized into processes. The resulting processes are, in turn, encapsulated into objects. Finally, the individual coordinates these individual mental constructs into a schema for a concept. In practice, of course, mental constructions rarely (if ever) occur in such a simple logical sequence. With respect to any given concept, an individual's early schema may be disorganized, incomplete, and contain inconsistencies. As she or he experiences *disequillibration* resulting from conflicts between expectations and results and engages in serious reflection, maturation as described by the APOS framework may takeplace.

It follows from the constructivist point of view in general and the APOS theoretical view in particular, that the role of the teacher is not to transfer her or his understanding of a concept to the students. Instead, the role of the teacher is to create situations in which students are likely to construct these actions, processes, objects and schemas for themselves. This is not to say that the students are expected to discover all, or even most, of the mathematics for themselves. Rather, a teacher implementing a pedagogical approach based on this theory would structure activities intended to provide students with a base of experience working with the actions, processes, and objects of mathematics in an attempt to help the student build elementary mental constructs and organize them into a coherent schema. [4]

Thus from the point of view of APOS Theory, the role of the mathematics education researcher begins with an attempt to identify the relevant cognitive structures which must be constructed in order to learn a given piece of mathematics. Such an attempt to model the cognition of an individual attempting to learn a mathematical concept is a daunting task. As noted in [26] by Steffe and Glasersfeld:

---

[3]Constructivism is the most widely accepted and influential contemporary learning theory in mathematics education. The APOS Theory described in this paper is a particular variation of constructivism which attempts to describe the nature and origin of the cognitive constructions made when mathematics is learned. See [9] for a particularly lucid introduction to the historical development, tenets, and pedagogical implications of constructivism.

[4]The reader interested in precalculus, calculus, discrete mathematics, or abstract algebra curricular materials designed specifically to help foster cognitive development of students with respect to the APOS constructions is referred to [4], [5], [7], or [20] .

> As with all general theoretical constructs, it is difficult to apply them to specific situations, when the cognizing subject is not ourselves but a 'subject' we are observing. In practice there may be observable behavioral indications, on the basis of which levels of abstraction can be determined, but making that determination is not simple. One might say that assuming something as 'given' or not is exclusively the subject's business. Hence, at best an observer can make educated guesses, taking into account – as does any experienced diagnostician – several indications collected over an extended period of observation.

It is the philosophical position of RUMEC that only after such an initial analysis is conducted should the researcher propose pedagogical innovations designed to help foster these cognitive constructions. Then the researcher should assess the effectiveness of these pedagogical strategies.

The goal of the current study was to begin this process of research and development by constructing theoretical analyses of the epistemology of three concepts: mean, standard deviation, and the Central Limit Theorem. That is, the goal was to produce three models of cognition - - specific mental constructions that a learner might make in order to develop her or his understanding of each of these concepts.

These models could serve several related purposes. Their descriptive nature and predictive usefulness could enable us to determine the extent to which the APOS epistemological framework is valuable in understanding the mental constructions made by students learning about elementary concepts in statistics. Such models could help to increase our understanding of how learning about mean, standard deviation, and the Central Limit Theorem might take place. They could serve as descriptive vehicles as we attempt to evaluate the extent to which the students can successfully perform mathematical tasks that require an understanding of these concepts. They may be used as the basis for designing improved curricular materials and pedagogical treatments. Finally, the development of such models can provide a base of information for future study on the epistemology and pedagogy associated with mean, standard deviation, and the Central Limit Theorem.

## 2 Research Methodology

Throughout the past decade, as more emphasis has been placed by the mathematics education research community on understanding cognitive development of students as they attempt to learn specific concepts, there has been a concurrent shift in research methodologies away from statistical studies involving control versus treatment groups toward more qualitative methods. One of the predominant methods of gathering qualitative data is the clinical interview. Romberg [21] discusses a variety of interview structures, which may range from informal discussions between researchers and participants to very structured conversations in which a predetermined list of questions is asked of each participant. [5]

For the present study, eight first-year students were randomly selected from those who received a grade of "A" in an elementary probability and statistics class during the fall semester of 1994 at a small state-supported liberal arts college. Across all eleven sections of this course, forty-five students out of 349 (about thirteen percent) received a grade of A. Only the 29 first-year students were considered as candidates for interviewing, since the study sought to determine the effect on the cognitive development of the students for whom the course was developed, rather than on the upper division students (including some mathematics majors) who elected to take this course much later in their academic careers. The eight students selected for interviewing represented three different instructors. It turned out that all of the students selected were female. This fact is not surprising given that 25 of the 29 first-year students receiving A's were, in fact, female.

---

[5]For an overview of trends in collegiate mathematics education research and related research methodologies, see Schoenfeld [22].

The course entitled “Statistical Decision Making" which these students had completed is very representative of introductory probability and statistics courses nationally. The first two weeks’ coverage under a typical syllabus included an overview of the role of statistics in society and an introduction to basic descriptive statistics, followed by approximately five weeks of probability including probability distributions for discrete random variables, the binomial distribution, standard and nonstandard normal distributions. Around the eighth week of class, students encountered the Central Limit Theorem and spent the remainder of the semester studying inferential statistics. Successful completion of this course satisfied the college’s general education mathematical literacy requirement, as would a course in “Functions and Graphs" or “Introduction to Computers". Although variations in emphasis existed across sections of this course, all sections used the same textbook (*Elementary Statistics, 5th edition*, Triola).

Once the students were selected (and agreed to participate in the study) interviews were conducted within the first six weeks of the Spring semester of 1995. The interviews centered around three core questions that were asked of each student.

1. What is meant by the word “mean" in statistics?
2. What is meant by “standard deviation"?
3. What is the Central Limit Theorem?

Students were given an opportunity to answer each question without prompting. Depending on their response, the interviewer would provide hints, clarification, and ask more questions. The interviewer attempted to maintain a relaxed atmosphere, asked students to give examples where they could, and to write anything that might help to explain their ideas. In this sense, the clinical interviews proceeded very much the same way questioning during a differential diagnosis of a medical patient might be done by a physician. First, a question is asked. Based on the response, and the interviewer’s knowledge of the subject matter, a follow up question is asked. This type of interchange continues until the interviewer believes that no more information will be forthcoming.

The goal of the first question was to put the students at ease. We naively believed that because these were the most successful students, they would have a strong object conception of mean which could be incorporated into schemas for more complex statistical concepts. Hence, the original intent of the interviews was to make progress in modeling students’ understanding of standard deviation, since experience with teaching statistics clearly indicates that this is a more difficult topic. It was a goal of question two to examine from the APOS theoretical framework why this might be the case. Finally, the Central Limit Theorem was chosen for study since it constitutes a crucial gateway between descriptive and inferential statistics.

All of the interviews were audio-taped and then transcripts of these sessions were produced to complement the record of written work which the student completed during the interview. The transcripts and written work were read carefully and analyzed in order to produce a list of mathematical issues that arose during the interviews. Focusing on these issues, we obtain results about the mental constructions that students appear to have made.

## 3 Results

We reiterate that the goal of the interviews was not to focus on mathematical performance per se, but rather to characterize in some meaningful way the conceptions of mean, standard deviation and the Central Limit Theorem which were held by the most successful students immediately after an introductory statistics course. Even so, in conducting the interviews (and subsequently reading the transcripts) one cannot help but be overwhelmed by the lack of understanding and by the heavy reliance (both conscious and unconscious) on algorithmic procedures which are not understood. That this might be the case for the students in the bottom half of the class is probably not surprising, but

that this is true for the “A" students is alarming. The following three points summarize the results seen in the interviews.

1. Although all eight students could correctly compute the mean of a set of numbers, when discussing specific examples many students confused the mean not only with the mode but also with population and sample proportions. In the terminology of Tall and Vinner, these concept images were often in conflict with the corresponding concept definition.
2. None of the eight students had a mature understanding of standard deviation.
3. None of the eight students interviewed had a working knowledge of the Central Limit Theorem.

We now examine each of these concepts through the lens of APOS Theory in order to more completely describe the students’ cognitive development.

## 3.1 Preliminary APOS Analysis: Mean

It is encouraging that none of the students interviewed were limited to an action conception of mean. Such a low level of cognitive development would have been indicated by an inability to calculate or describe the process of finding the mean in the absence of the formula. All eight students demonstrated that they had moved beyond an action conception to a process conception of mean. They had internalized the notion of numbers coming in, a certain computation taking place, and a number being output. The excerpts below from the interviews with Fran and Penny[6] illustrate a strong sense that the mean IS a process of computation. When asked what is meant by mean, they said:

> [Penny:] An average. Well, the mean is when you would add all the numbers involved and divide by that number, and that would give you the mean.

> [Fran:] It’s...You have a certain amount of numbers, and you take all of those numbers, and add them together, and divide by the number of numbers to get the mean.

The above descriptions, of course, are not incorrect. Rather, they indicate at least a process conception of mean because the computational algorithm is clearly under the internal control of each of these students. But upon repeated probing for what the mean represents, the students invariably returned to a description of how to calculate the mean. When asked why or how they might use the mean their responses, like those of Beth and Paula, indicate a reliance on the formula and an obvious confusion concerning what is meant by mean.

> [Paula:] Umm...It was just something we calculated. I..I.. honestly don’t know why. You used it, basically, throughout the entire course to compute the others. It was something that was taught to us by the professor, and then we didn’t really use it except for the test, and that was it.

> [Beth:] The average. I don’t know. I guess it’s just the average number. I don’t know how to explain it. The mean, to me, would be like the middle, maybe that’s the mode, the middle number of like all of these. I am not really sure how to explain what it would mean. I just know how to do the formula.

For these students, the mean is neither a measure of central tendency, an object with properties, nor a cognitive entity. It is a process only. This lack of object conception is best exemplified in the following exchange between the interviewer and Fran.

> [Interviewer:] Let’s suppose you took four quizzes, each had ten possible points, and you got scores of 10, 5, 6, and 7. ............and you calculated a mean of what?

> [Fran:] 9.5

> [Interviewer:] Does that strike you as reasonable?

> [Fran:] Yeah, somewhat.

---

[6]The names of the students have been changed to protect their privacy.

The interviewer tried, without success, to help Fran to see the inconsistency of this result with the data set it was supposed to represent. Although “number" is an object to Fran, the mean is not. Put another way, even though the output of the calculation is a number, the mean to Fran is something you calculate, not an object with properties she could invoke in order to determine whether her calculation makes sense.

## 3.2 Preliminary APOS Analysis: Standard Deviation

Of the eight students interviewed, three demonstrated the lowest level of understanding of standard deviation–an action conception. At this cognitive level, standard deviation is only a rule or a formula to be given and followed, and students are unable to describe the algorithm. Representative excerpts of the transcripts from these three students can be seen by the comments of Beth and Fran below.

> [Beth:] Oh Gosh, standard deviation is, I'm not sure really why you do it exactly. I just know...I don't even know if I can remember the formula but the way I was taught was just like given a formula and I can work out a problem.

Even with repeated probing for a description or interpretation of standard deviation or its formula, the only thing Fran could say was:

> [Fran:] I figured it out, but I could never remember what the purpose of it was.

Given Fran's action conception of standard deviation, along with a process conception of mean, it is not surprising that she doesn't understand why people study statistics. In APOS terminology, Fran has a very weak “statistics schema."

In sharp contrast to the situation for mean, none of the students interviewed demonstrated an *appropriate* process conception of standard deviation. This could be caused by the fact that most (if not all) of the students had been exposed to the concept of mean before this statistics course but had not seen standard deviation previously. Additionally, the formulas used to compute the standard deviation are obviously more complicated, and their use is often unjustified in the students' minds. In any case, the lack of a correct process conception of standard deviation indicates the ineffectiveness of the standard pedagogical treatment of this topic for these students.

Recall that according to the APOS framework, objects are constructed by encapsulating processes. Thus, since none of the students interviewed had an appropriate process conception of standard deviation, none could demonstrate an appropriate object conception. It is interesting to observe that of the five studentswho had progressed past an action conception of standard deviation, every one had one of the following two inappropriate object conceptions.

Two of the five students had developed inappropriate object conceptions of standard deviations as representing the distance between members of an ordered set. Penny correctly believed that the standard deviation of the set consisting of {2,3,4} was 1, but upon probing it became evident that she was measuring the distance between members of the set. When asked about the standard deviation of the set consisting of {1,2,6}, she determined that it would be 2.5–which she obtained by averaging the numbers 1 and 4. Similarly, Amy thought the standard deviation of {2,4,6,8,10} was 2, and the standard deviation of {2,3,7,8,10} was approximately 2.5.

The remaining three students had inappropriate object conceptions of standard deviation as tied to single data points. This can be seen in interchanges with Cindy and Rachel. When asked what standard deviation represented, Cindy replied:

> [Cindy:] How far apart a certain piece of data falls from the mean. How far away it falls. If you had a certain test grade or a certain quiz grade of 8, you could find the standard deviation of how far off from the mean this grade was.

Similarly, Rachel said:

> [Rachel:] Standard deviation has to do with the normal distribution and how many...how far away from the mean of a certain number stands.

These two inappropriate object conceptions of standard deviation are clearly the result of encapsulating the wrong process. In the first case, the process could be described as putting the numbers in order and measuring the distance between them (with some possible further step if the distance is not uniform). In the second case, again the process is measurement of a distance, but this time one first selects a particular number and measures its distance to the mean. (Of course, this process is actually a part of the Z-score computation, and the missing element is division by the standard deviation.) This process appears to result in an inability to distinguish between Z-scores and the standard deviation as a measure of spread. In either case, when students encapsulate the wrong process, they produce the wrong cognitive object.

## 3.3 Preliminary APOS Analysis: The Central Limit Theorem

The cognitive constructions which are necessary to understand statements of theorems, methods of proofs, and other mathematical concepts which are more complex than individual constructions may be modeled through the use of schemas. In general, modeling these schemas may be quite involved.

To understand the Central Limit Theorem one must understand the distinction between a population and a sample. An individual sample must be a cognitive object in a collection of all possible samples. The mean for each sample must also be an object, and so must the notion of “distribution". Before gathering the interview data, it was conjectured that understanding the Central Limit Theorem would involve coordinating in a very precise way the two mathematical objects of population mean and sample mean with the two additional objects of population standard deviation and sample standard deviation.

However none of these eight students was able to discuss the Central Limit Theorem in a meaningful way. Paula, for example, could only vaguely recall that some formula was involved:

> [Paula:] You’re given thirty, and you want a sample of 25 or something like that, and um $x$ minus $\bar{x}$ over the ... I can’t... It was just this $\mu$ over $x$... I can’t remember. Like I know it’s not $x$ over $\bar{x}$ for the standard deviation and your $x$ is for like...This might even be the wrong formula which it would ...I vaguely remember what it was, and I’d remember if I would see it.

It is perhaps worth noting that one of the instructors whose students participated in the interviews reports placing major emphasis on the idea of the Central Limit Theorem. We saw no evidence that students from this section had progressed any further in their cognitive development relative to this topic than the students of the other instructors. We see in excerpts from Fran and Beth that some of the students even felt the Central Limit Theorem was not an important part of their course.

> [Beth:] I know that...(pause) I seem to remember that I was talking about Central Limit Theorem around the same time we talked about normal distributions, but I don’t really remember exactly what we used it for, and it might have just been. I just remember talking about it, but I remember it not being an important formula as one that we used more often.

> [Fran:] We didn’t go over that as I recall.

Since none of the eight students interviewed demonstrated an understanding of the statement of the Central Limit Theorem, no progress was made toward refining the preliminary APOS analysis which we made from the theory alone.

# 4 Conclusion

In terms of the performance data, one must ask whether or not these results are replicable. Are they representative of outcomes from this type of course nationally? Although more research is necessary

before any conclusion about performance can be drawn, the low level of cognitive development resulting from this course seems to agree with that outlined in [8].

In terms of mental constructions, the action-process-object epistemological framework served as a very useful tool for describing student understanding of mean and standard deviation. The variance between the level of cognitive development which resulted in “success" for these students and mature understanding of these concepts is particularly striking. Viewing student understanding in terms of APOS suggests that these students will continue to have cognitive difficulties with all of the topics in inferential statistics until such time as they develop appropriate object conceptions of mean and standard deviation. Only after developing rich object conceptions, will they have the essential building blocks necessary for constructing a viable schema for the Central Limit Theorem.

Three additional statements seem apparent from the interviews as well. First, although current reform literature [15], [17], [30] has apparently converged on the consensus view that elementary statistics teaching should be set in the context of the real-world, the plethora of real world examples in the textbook used by these students did not appear to have the desired effect on their cognitive development relative to statistical concepts. Only three of the eight students interviewed were able to give a reasonable example of mean other than the average of test grades in a course. This observation indicates that something more closely related to pedagogical innovation rather than content innovation may be necessary to improve statistical understanding.

Second, in reading the transcripts of the interviews, one is struck by the observation that data sets are apparently not objects for even the most successful students of statistics. Notice that neither Penny, Fran, Paula, nor Beth used the word “set" when describing what is meant by mean. Something deeper than an imprecise use of language seems to be going on here. Clearly the idea of “number" is an object to every one of the students interviewed, but sets of numerical data, be they sets representing populations or samples, often do not appear to have the status of cognitive object.

Third, even students who have been completely successful in an elementary statistics course may not have much of an understanding of why one would study statistics, or of what the body of knowledge and methods of statistical practice are really all about.

The usefulness of APOS Theory in the context of this study is indicated not only by the powerful descriptive tools it provides, but also by the fact that this analysis raised questions even deeper than the original ones. The following is a list of questions which one might pursue with this study as a foundation.

1. In what way are functions (such as the standard deviation) whose inputs are sets, easier or more difficult to conceptualize than functions of a single real variable?
2. What pedagogical strategy can promote the interiorization of the correct process for standard deviation?
3. What specific activities can promote the encapsulation of a correct process conception of standard deviation into an object?
4. More generally, what specific activities can promote the encapsulation of a function whose domain is a set of sets?
5. What specific activities can promote the formation of a set as a cognitive object?
6. Do the cognititve constructions proposed by the APOS Theory (for example the object conceptions of mean and standard deviation) actually exist?
7. What, if any, other cognitive structures beyond object conceptions of sample, population, mean, distribution, and standard deviation are necessary for a viable Central Limit Theorem schema?
8. What do students believe the nature of statistics to be?

# 5 Bibliography

## References


[1] Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. *Research in Collegiate Mathematics Education II.* Conference Board of the Mathematical Sciences: Issues in Mathematics Education. *Vol. 6*, 1–32.

[2] Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*. *23*, 247–285.

[3] Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (in press). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of Mathematical Behavior*.

[4] Dubinsky, E. & Leron, U. (1994). *Learning Abstract Algebra with ISETL.* New York: Springer-Verlag.

[5] Fenton, W. & Dubinsky, E. (1996). *Learning Discrete Mathematics with ISETL.* New York: Springer-Verlag.

[6] Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, *27*, 267–305.

[7] Dubinsky, E., Schwingendorf, K. E., & Mathews, D. M. (1995). *Calculus, Concepts and Computers* (2nd ed.). New York: McGraw-Hill.

[8] Garfield, J. & Ahlgren, A. (1988). Difficulties in learning basic concepts in probability and statistics: Implications for research. *Journal for Research in Mathematics Education*, *19(1)*, 44–63.

[9] Glasersfeld, E. von. (1995). *Radical constructivism: A way of knowing and learning.* London: Falmer Press.

[10] Gray, Eddie & Tall, David (1991). Duality, Ambiguity, and Flexibility in Successful Mathematical Thinking. In F. Furinghetti (Ed.) *Proceedings of the 15th Annual Conference of the International Group for the Psychology of Mathematics Education* Volume 2, (72–79). Assisi, Italy.

[11] Hardiman, P.T. (1984). Usefulness of a balance model in understanding the mean. *Journal of Educational Psychology*. *Vol. 76*, 792-801.

[12] G. Harel & Dubinsky, E. (Eds.). (1992). *The concept of function: Aspects of epistemology and pedagogy, MAA notes 25*. Washington, DC: Mathematical Association of America.

[13] Hong, E. & O'Niel, H.F. (1992). Instructional strategies to help learners build relevant mental models in inferential statistics. *Journal of Educational Psychology. Vol. 84, No. 2.*, 150-159.

[14] Kilpatrick, J. (1978). Variables and methodologies in research on problem solving. In L. Hatfield (Ed.), *Mathematical Problem Solving*. ERIC, Columbus, OH, 7–20.

[15] Landwehr, J., Swift, J., & Watkins, A. (1987). *Exploring surveys and information from samples*. Palo Alto, California: Dale Seymour Publications.

[16] Leron, U., & Dubinsky, E. (1995). An abstract algebra story. *American Mathematical Monthly, 102*, 227-242.

[17] Li, K. & Shen, S. (1992). Students' weaknesses in statistical projects. *Teaching Statistics, 14(1)*, 2–8.

[18] Mokros, J. & Russell, S. (1995). Children's Concept of Average and Representativeness. *Journal for Research in Mathematics Education*, *Vol. 26, No. 1*, 20–39.

[19] Piaget, J. & Garcia, R. (1989). *Psychogenesis and the History of Science* (H. Feider, Trans.). New York: Columbia University Press. (Original work published 1983).

[20] Reynolds, B, Przybylski, J., Kiaie, C., Schwingendorf, K. , & Dubinsky, E. (1996). *Precalculus, Concepts and Computers*. New York: McGraw-Hill.

[21] Romberg, T. (1992). Perspectives on scholarship and research methods. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*. (49–64). New York: Macmillan.

[22] Schoenfeld, A. (1994). Some Notes on the Enterprise (Research in Collegiate Mathematics Education, That Is). In *Research in Collegiate Mathematics Education I*. Conference Board of the Mathematical Sciences: Issues in mathematics Education. *Vol. 4*, 1–19.

[23] Sfard, A., (1987). Two conceptions of mathematical notions, operational and structural. In A. Borbàs (Ed.), *Proceedings of the 11th Annual Conference of the International Group for the Psychology of Mathematics Education* (162–169). Montreal: University of Montreal.

[24] Sfard, A., and L. Linchevski, (1996).The gains and the pitfalls of reification: The case of algebra *Educational Studies in Mathematics*, (in press).

[25] Sfard, A., (1991). On the dual nature of mathematical conceptions. *Educational Studies in Mathematics*. *22*, 1–36.

[26] Steffe, L. & Glassersfeld, E. (1988). On the construction of the counting scheme. In *Construction of Arithmetical Meanings and Strategies*. New York, Springer.

[27] Tall, D. & Vinner, S. (1981). Concept image and concept definition in Mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*. *12*, 151–169.

[28] Thompson, P. (1994). Students, Functions, and the Undergraduate Curriculum. In *Research in Collegiate Mathematics Education I*. Conference Board of the Mathematical Sciences: Issues in Mathematics Education, *Vol. 4*, 21–44.

[29] Thurston, W. (1994). On proof and progress in mathematics, *Bulletin (New Series) of the American Mathematical Society*. *Vol. 30, No. 2*, 161-177.

[30] Velleman, P. & Hoaglin, D. (1992). Data Analysis. In D.C. Hoaglin and D. Moore (Eds.). *Perspectives on contemporary statistics, MAA notes 21* (19–39). Washington, DC: Mathematical Association of America.

[31] Vinner, S. (1992). The function concept as a prototype for problems in mathematics learning. In G. Harel and E. Dubinsky (Eds.). *The concept of function: Aspects of epistemology and pedagogy, MAA notes 25* (195–213). Washington, DC: Mathematical Association of America.

## Acknowledgments

The authors are extremely grateful to three groups of people for their contributions. First, to all of the researchers whose works were cited in this paper. Second, to the members of the Research in Undergraduate Mathematics Education Community (RUMEC) for their thoughtful comments and criticisms of earlier drafts of this manuscript. Finally, a special thanks goes to the students who voluntarily participated in the clinical interviews reported herein.